Conjecture + Proof = Theorem

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Agenda

- Conjectures, proofs and theorems
- Contest
  - Conjecture or theorem?
- Answers to the contest
- Chocolate Bar Theorem
- Win $1,000,000 for proving a conjecture!

Theorems vs. conjectures

- Mathematicians try to prove hypotheses and discoveries using logic
  - <u>Theorems</u> are important mathematical facts that have been proven to be true
  - <u>Conjectures</u> are mathematical discoveries that seem to be true, but which haven’t been proven

What is a proof?

- A proof demonstrates why the theorem is true
  - It’s a chain of logical reasoning consisting of many steps that are nearly obvious
  - A proof assumes certain self-evident facts (axioms) and may build upon theorems that have been proven already
- Conjecture + Proof = Theorem

Let's prove the following

- 91 is not prime
- 7 is prime

91 is not a prime number

Proof:

- A prime number is a positive integer that is greater than 1 and that is divisible only by 1 and itself
- 91 is divisible by 13 and 7, since 13 x 7 = 91
- Therefore, 91 is not a prime number

7 is a prime number

Proof:

- 7 is divisible by only 1 and itself
  - $7 \div 2 = 3$ R 1
  - $7 \div 3 = 2$ R 1
  - $7 \div 4 = 1$ R 3
  - $7 \div 5 = 1$ R 2
  - $7 \div 6 = 1$ R 1
- Therefore, 7 is a prime number

Contest

- Divide up into teams of four
- I'll present several math discoveries
  - Some are proven theorems
  - Some are unproven conjectures
- Your team must decide which are proven theorems and which are unproven conjectures
- The winning team will win prizes

1. Conjecture or theorem?

- There are infinitely many prime numbers
  - That is, there is no largest prime number

2. Conjecture or theorem?

- There are an infinite number of <u>twin primes</u>
  - Two numbers are twin primes if they are both prime numbers and they differ by 2
- The following are twin primes:
  - 3 and 5
  - 5 and 7
  - 11 and 13
- Try to find another pair of twin primes

3. Conjecture or theorem?

- Every even number greater than 2 is the sum of two prime numbers
- For example:
  - 4 = 2 + 2
  - 6 = 3 + 3
  - 24 = 11 + 13
  - 100 = 17 + 83
- Find 2 prime numbers whose sum is 30

4. Conjecture or theorem?

- Every positive integer is the sum of four squares
- Examples:
  - $10 = 2^2 + 2^2 + 1^2 + 1^2 = 4 + 4 + 1 + 1$
  - $16 = 4^2 + 0^2 + 0^2 + 0^2 = 16 + 0 + 0 + 0$
  - $50 = 6^2 + 3^2 + 2^2 + 1^2 = 36 + 9 + 4 + 1$
  - $100 = 9^2 + 3^2 + 3^2 + 1^2 = 81 + 9 + 9 + 1$
- Find 4 squares whose sum is 21

5. Collatz Sequences

- Choose a positive integer and create a list of integers as follows:
  - If the current number is even, then divide it in half to get the next number
  - If the current number is odd, then multiply it by 3 and add 1 to get the next number

5. Example Collatz Sequences

- 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, …
- 15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, …
- 321, 964, 482, 241, 724, 362, 181, 544, 272, 136, 68, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, …

5. Conjecture or theorem?

- Every Collatz Sequence ends with the numbers “4, 2, 1,” repeating
  - 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, **4, 2, 1, 4, 2, 1, …**
  - 321, 964, 482, 241, 724, 362, 181, 544, 272, 136, 68, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, **4, 2, 1, 4, 2, 1, …**
- What’s the sequence that starts with 6?

6. Fibonacci numbers

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
- Each number in the sequence is equal to the sum of the previous two numbers
  - 1 = 1 + 0
  - 2 = 1 + 1
  - 3 = 2 + 1
  - 5 = 3 + 2
  - 8 = 5 + 3

6. Conjecture or theorem?

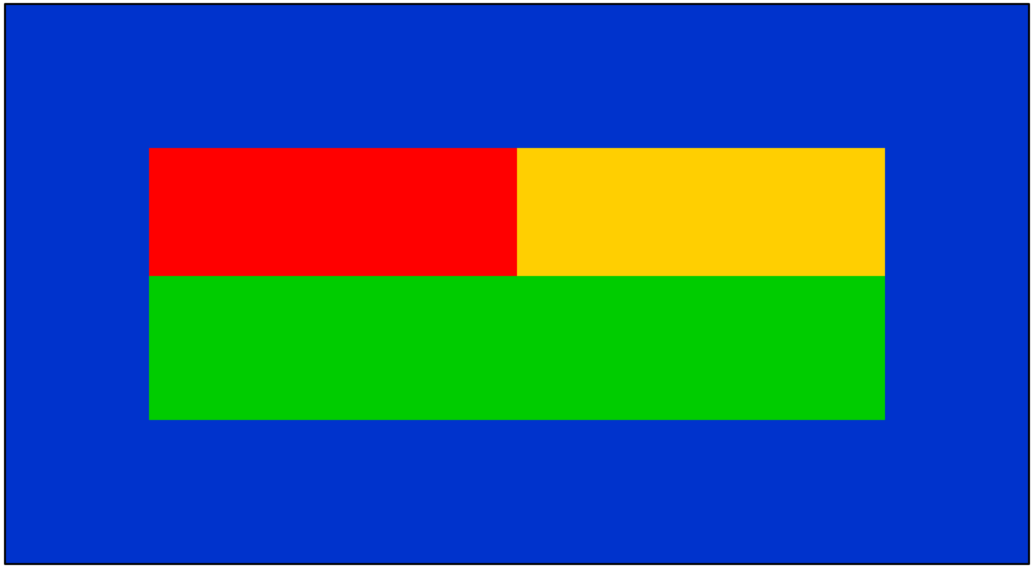
- Every positive integer is the sum of two or more different Fibonacci numbers
- For example:
  - 10 = 8 + 2
  - 50 = 34 + 13 + 3
  - 100 = 89 + 8 + 3
- Find the distinct Fibonacci numbers whose sum is 33. Here are the Fibonacci numbers:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

7. Conjecture or theorem?

- Every map can be colored using 4 colors
  - No two adjacent regions have the same color
  - Adjacent regions share a border, not just a point
- First discussed in 1852 by Francis Guthrie
  - He noticed that only 4 colors were needed to color a map of the counties of England

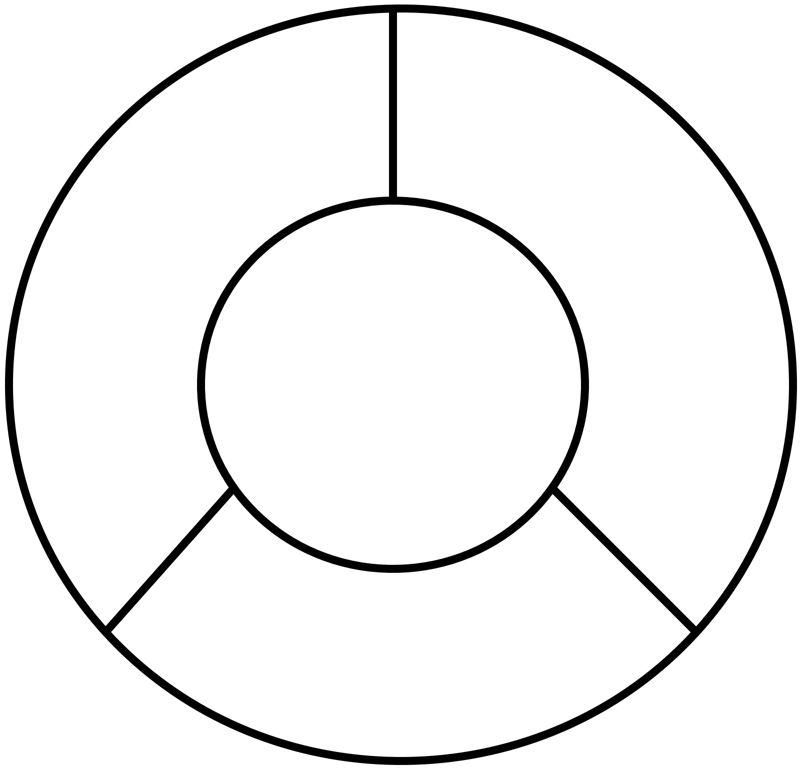
7. Conjecture or theorem?

- The following map can be colored with 4 colors, but not with 3 colors



7. Conjecture or theorem?

- Try to color this map with 4 colors



End of contest

- Please hand in your answers

1. Infinitude of Primes Theorem

- Euclid’s Second Theorem
  - The number of primes is infinite
  - Proved by Euclid of Alexandria over 2,300 years ago!
- The proof is fairly simple
  - I learned it in high school

2. Twin Prime Conjecture

- There are infinitely many twin primes
- First proposed by Euclid
  - For over 2,300 years, mathematicians have been unable to prove it
- Largest twin primes (so far):

  $16{,}869{,}987{,}339{,}975 \times 2^{171{,}960} + 1$ and
  $16{,}869{,}987{,}339{,}975 \times 2^{171{,}960} + 3$

3. The Goldbach Conjecture

- Every even number greater than 2 is the sum of two prime numbers
- Proposed by Goldbach and Euler in 1742
  - Euler, one of the greatest mathematicians of all time, could not prove it
- Computers have verified the conjecture for numbers up to 200,000,000,000,000,000

4. Four-Square Theorem

- Every positive integer is the sum of four squares
- The ancient Greek mathematician Diophantus knew about it
- Proven by Joseph Louis Lagrange in 1770

5. Collatz Conjecture

- All Collatz sequences end with **4, 2, 1,** repeating forever
- First proposed by Lothar Collatz in 1937
  - Not yet proven
- The conjecture is true for all start values up to 27,021,597,764,222,976
  - Checked by computer

6. Zeckendorf's Theorem

- The theorem implies that every positive integer is the sum of two or more different Fibonacci numbers
- Edouard Zeckendorf was an amateur mathematician
- He proved the theorem in 1939

7. Four Color Theorem

- Every map can be colored with 4 colors
- Proved by Kenneth Appel and Wolfgang Haken in 1976
  - The proof required a super computer
    - The computer spent 1200 hours checking many specific maps
  - It took many years to check the proof for errors

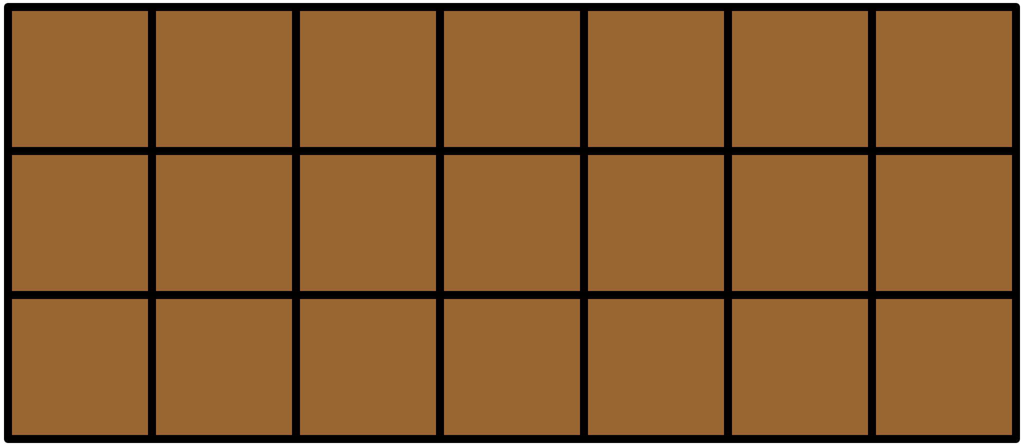
Summary

1. Infinitude of Primes Theorem
2. Twin Prime Conjecture
3. Goldbach’s Conjecture
4. Lagrange’s Four-Square Theorem
5. Collatz Conjecture
6. Zeckendorf’s Theorem
7. Four Color Theorem

Chocolate Bar Theorem

- If a chocolate bar consists of N squares, then it takes N-1 splits to break it up into the N squares
  - For example, if the bar consists of 20 squares, then 19 splits are required to break up the bar into the 20 squares
  - Each split breaks one piece along a horizontal or vertical line

Chocolate Bar Theorem



- For example, this chocolate bar consists of 21 squares
- Therefore, 20 splits are required to break up the bar into the 21 squares

Chocolate Bar Theorem

- Proof:
  - Each time you split a piece, the total number of pieces increases by 1
  - After the first split, you have two pieces
  - After the second split, you have three pieces
  - After N-1 splits, you have N pieces
  - Therefore, if the bar consists of N squares, then N-1 splits are required

Chocolate Bar Theorem

- Let's test the theorem with real chocolate bars
- Yummy!

Win $1,000,000 for proving a conjecture

- In May 2000, the Clay Mathematics Institute offered seven $1,000,000 prizes for the solutions to each of seven unsolved problems in mathematics
  - Most of the these Millennium Problems are conjectures
  - One has been proven recently!

References

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- Proofs: http://mathforum.org/dr.math/faq/faq.proof.html
- Conjectures: http://en.wikipedia.org/wiki/Conjecture
- Lagrange's four-square theorem: http://en.wikipedia.org/wiki/Lagrange's_four-square_theorem
- Ancient Greek Mathematicians: http://encarta.msn.com/encyclopedia_761578291_6/Mathematics.html

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- No largest prime: http://mathforum.org/library/drmath/view/57092.html
- Fibonacci Numbers: http://en.wikipedia.org/wiki/Fibonacci_number
- Zeckendorf’s Theorem: http://en.wikipedia.org/wiki/Zeckendorf’s_theorem
- Breaking chocolate bars: http://www.cut-the-knot.org/proofs/chocolad.shtml
- Millennium Problems: http://www.claymath.org/millennium/